Math 4200
Wednesday October 21
2.5 discuss conformal diffeomorphisms of the disk via the maximum modulus principle, in Monday's notes, and the Poisson integral formula for harmonic functions in the disk, in today's notes. On Friday we will begin Chapter 3 about series representations of analytic functions.

Announcements: Quiz today!

Warm-up exercise:

Application to harmonic function theory (in partial differential equations). There is an analog of the Cauchy integral formula for harmonic functions, that expresses the value of a harmonic function inside a domain in terms of an integral over the boundary which uses the harmonic function's boundary values. It's much messier to write down than the Cauchy integral formula in general - if you wanted to take the real part of the Cauchy integral formula you'd also need to know the boundary values of the conjugate to the harmonic function, to deduce the values of the harmonic function in the interior, so you can't just use the CIF, like we did for the mean value property. In the case where the domain is the unit disk (or a scaled disk), this analog to the CIF is known as the *Poisson integral formula* and we can prove it via the mean value property and Mobius transformations.

Theorem (Poisson integral formula for the unit disk) Let $u \in C^2(\mathrm{D}(0;1)) \cap C(\bar{\mathrm{D}}(0;1))$, and let u be harmonic in $\mathrm{D}(0;1)$. Then the Poisson integral formula recovers the values of u inside the disk, from the boundary values. It may be expressed equivalently in complex form or real form. For $z_0 = x_0 + i\,y_0 = r\,e^{i\,\varphi}$ with $|z_0| < 1$,

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z_0|^2}{|z_0 - e^{i\theta}|^2}\, u(e^{i\theta})\,d\theta$$

$$u(r\cos\varphi, r\sin\varphi) = \frac{1}{2\pi}\int_0^{\pi} \frac{1-r^2}{r^2 - 2r\cos(\theta-\varphi) + 1} u(\cos(\theta), \sin(\theta))\,d\theta$$

$*$ First, check why the CIF formula wouldn't work directly (except for $u(0)$) unless we knew the harmonic conjugate.

$**$ But we do know the mean value property, and we can combine this with the Mobius transformations in yesterday's notes! (Actually we only know the mean value property if u is harmonic on a slightly larger disk than $D(0; 1)$, but it also holds for harmonic $u \in C^2(D(0; 1)) \cap C(\bar{D}(0; 1))$, by a rescaling, limiting process.) In any case, consider the Mobius transformation

$$g_{z_0}(z) = \frac{z_0 + z}{1 + \bar{z_0}\, z}$$

Then $u\left(g_{z_0}(z)\right)$ is harmonic on the unit disk (do you remember why, from a Chapter 1 homework problem?). So by the mean value property for the composition,

$$u(z_0) = u\left(g_{z_0}(0)\right) = \frac{1}{2\pi}\int_0^{2\pi} u\left(g_{z_0}\left(e^{i\alpha}\right)\right) d\alpha$$

Now we just change variables, and after some computations out pops the Poisson integral formula! Consider α as a function of θ on the unit circle via

$$g_{z_0}\left(e^{i\alpha}\right) = e^{i\theta}$$

$$g_{-z_0}\left(e^{i\theta}\right) = e^{i\alpha}$$

So,

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(g_{z_0}\left(e^{i\alpha}\right)\right) d\alpha$$

$$= \frac{1}{2\pi}\int_0^{2\pi} u\left(e^{i\theta}\right)\alpha'(\theta)\, d\theta.$$

To get $\alpha'(\theta)$ we differentiate e.g. the second change of variables formula, using the chain rule for curves and regular Calculus

$$\frac{d}{d\theta}\left(g_{-z_0}\left(e^{i\theta}\right)\right) = \frac{d}{d\theta}\left(e^{i\alpha}\right)$$

$$g_{-z_0}{}'\left(e^{i\theta}\right) i\, e^{i\theta} = i\, e^{i\alpha}\, \alpha'(\theta).$$

From yesterday's notes, $g_{-z_0}{}'(z) = \dfrac{1 - |z_0|^2}{\left(1 - \bar{z_0}\, z\right)^2}$ so the identity above for $\alpha'(\theta)$ reads

$$\frac{1 - |z_0|^2}{\left(1 - \bar{z_0}\, e^{i\theta}\right)^2}\, i\, e^{i\theta} = i\, \frac{-z_0 + e^{i\theta}}{1 - \bar{z_0}\, e^{i\theta}}\, \alpha'(\theta)$$

so (repeating some equations on this page hoping for lecture clarity):

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(g_{z_0}\left(e^{i\alpha}\right)\right)d\alpha$$

$$g_{z_0}\left(e^{i\alpha}\right) = e^{i\theta}, \quad g_{-z_0}\left(e^{i\theta}\right) = e^{i\alpha}$$

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(e^{i\theta}\right)\alpha'(\theta)\,d\theta$$

where $\alpha'(\theta)$ satisfies the identity

$$\frac{1-|z_0|^2}{\left(1-\bar{z}_0\, e^{i\theta}\right)^2}\, i\, e^{i\theta} = i\,\frac{-z_0+e^{i\theta}}{1-\bar{z}_0\, e^{i\theta}}\,\alpha'(\theta)$$

$$\frac{1-|z_0|^2}{\left(1-\bar{z}_0\, e^{i\theta}\right)}\,\frac{e^{i\theta}}{-z_0+e^{i\theta}} = \alpha'(\theta)$$

$$\frac{1-|z_0|^2}{\left(1-\bar{z}_0\, e^{i\theta}\right)\left(1-z_0 e^{-i\theta}\right)} = \alpha'(\theta)$$

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z_0|^2}{\left|z_0-e^{i\theta}\right|^2}\,u\left(e^{i\theta}\right)d\theta\ !$$

QED!!

Harmonic functions exist and are uniquely determined by their boundary values - we know that from the maximum principle for continous boundary values, and it's even true if the boundary values are only piecewise continuous....in the disk the harmonic functions can be expressed using Fourier series, or with the Poisson integral formula we just proved, and as we've mentioned, they describe various physical phenomena, such as equilibrium temperature distributions in 2-dimensional plates having controlled boundary temperatures....also related to random walk phenomena in probability, other applications.

http://mathfaculty.fullerton.edu/mathews/c2003/DirichletProblemDiskMod.html

Extra Example 1. Find the function $u(x, y) = u(r\cos\theta, r\sin\theta)$ that is harmonic in the unit disk $D_1(0) = \{z : |z| < 1\}$,

and takes on the boundary values $u(\cos\theta, \sin\theta) = U(\theta) = \begin{cases} 1, & \text{for } \frac{\pi}{2} < t < \pi, \\ 0, & \text{for } -\frac{\pi}{2} < t < \frac{\pi}{2}, \\ -1, & \text{for } -\pi < t < -\frac{\pi}{2}. \end{cases}$

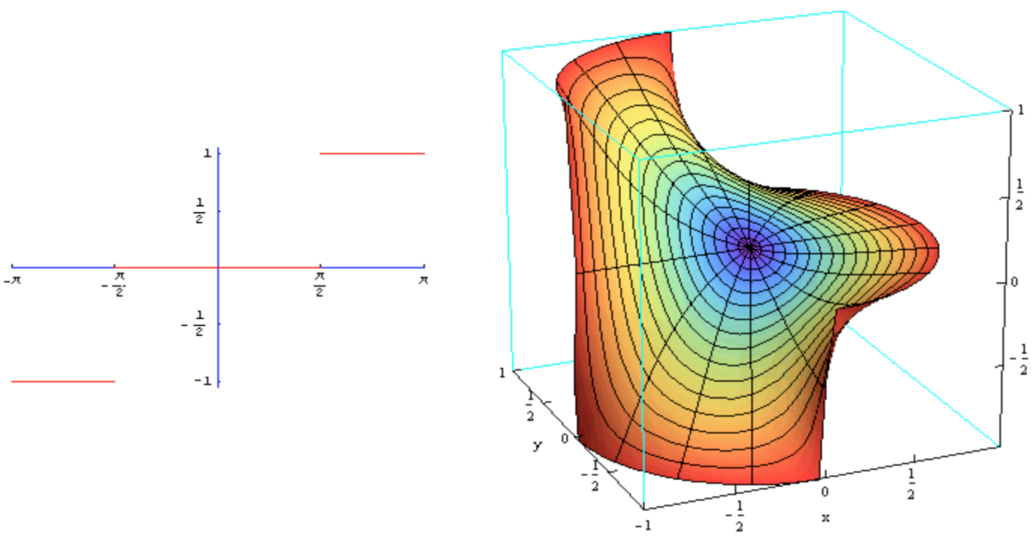

Figure 1. The graphs of $U(\theta) = u(\cos\theta, \sin\theta)$ and $u(x, y) = u(r\cos\theta, r\sin\theta)$.

warmup for Chapter 3, if we have time today....

Chapter 3: Series representations for analytic functions. Section 3.1: Sequences and series of analytic functions.

Recall a key analysis theorem which we proved and used in our discussion of uniform limits of analytic functions last week, in which we used Morera's theorem to prove that uniform limits of analytic functions are analytic:

Theorem Let $A \subseteq \mathbb{C}, f_n : A \rightarrow \mathbb{C}$ continuous, $n = 1, 2, 3 \ldots$ If $\{f_n\} \rightarrow f$ *uniformly*, then f is continuous. (The same proof would've worked for $A \subseteq \mathbb{R}^k$, $F_n : A \rightarrow \mathbb{R}^p$, $\{F_n\} \rightarrow F$ uniformly.)

Corollary Let $A \subseteq \mathbb{C}, f_n : A \rightarrow \mathbb{C}$ continuous, $n = 1, 2, 3 \ldots$ If $\{f_n\}$ is *uniformly Cauchy*, then there exist a continous limit function $f : A \rightarrow \mathbb{C}$, with $\{f_n\} \rightarrow f$ uniformly.